Urok nierówności Markowa

W. Pleśniak

Jagiellonian University of Cracow, Institute of Mathematics

Stanisława Łojasiewicza 6, 30-348 Kraków, Poland

Wieslaw.Plesniak@im.uj.edu.pl

Abstract

We present a survey on recent research on the multivariate Markov inequality. We illustrate the power of this inequality by giving a number of its applications in the theory of extension and polynomial approximation of $\mathcal{C}^\infty$ functions defined on compact subsets of $\mathbb{R}^n$

1 Markov Inequality

In 1889, A.A. Markov answered a question posed two years earlier by Mendeleev by showing that for every polynomial p in one variable

$$|p'(x)| \leq (\deg p)^2 \|p\|_{[-1,1]}, \quad \text{as } x \in [-1,1], \tag{MI_1}$$

where $\|p\|_I = \sup |p|(I)$. This result is best possible since for the Chebyshev polynomials $T_k(x) = \cos k \arccos x$ $(x \in [-1,1])$, $k = 1, 2, \ldots,$ of degree k one has $\|T_k\|_{[-1,1]} = 1$ and $|T_n'(\pm 1)| = n^2$.

Markov's inequality became soon a fascinating object of investigations. The reason lay with its numerous applications in different domains of mathematics and physics. A corresponding theory in the several variables case is relatively new and until the late 1970's all known extensions of Markov's inequality dealt practically with the case where the line-segment in (MI_1) is

Key words and phrases: Multivariate Markov inequality, Bernstein-type approximation of $\mathcal{C}^\infty$ functions, Whitney jets, continuous linear extension of $\mathcal{C}^\infty$ functions, Jackson-type inequality

replaced by a convex compact subset of $\mathbb{R}^n$ with non-void interior. One of the obstacles was the fact that for some cuspidal sets in $\mathbb{R}^n$ no multivariate counterpart of (MI_1) can be proved. A simple example was first given by Zerner 1969.

Example 1.1 Let $E = \{(x,y) \in \mathbb{R}^2 : 0 < y \leq \exp(-1/x), 0 < x \leq 1\} \cup \{(0,0)\}$ and let $P_k(x,y) = y(1-x)^k$ for $k = 1, 2 \ldots$. Then $\deg P_k = k+1$, $\|\partial P_k/\partial y\|_E = 1$, while $\|P_k\|_E < \exp(-\sqrt{k})$ for $k = 1, 2, \ldots$, and therefore there are no constants $M > 0$ and $r > 0$ such that for each k,

$$\|\partial P_k/\partial y\|_E \leq M(k+1)^r \|P_k\|_E.$$

In the sequel, a compact subset of $\mathbb{R}^n$ is said to preserve (or admit) *Markov's inequality*, or simply to be *Markov*, if there exist constants $M > 0$ and $r > 0$ such that for each polynomial p in $\mathbb{R}^n$ we have

$$\|\operatorname{grad} p\|_E \leq M(\deg p)^r \|p\|_E. \qquad (MI_n)$$

A satisfactory theory of the multivariate Markov inequality was developed in the last 15 years by W. Pawłucki and W. Pleśniak, P. Goetgheluck, M. Baran, A. Jonsson, J. Siciak, A. Zeriahi, L. Bos and P.D. Milman, A. Goncharov and others.

The goal of this talk is to present a state-of-the-art survey of investigations concerning the inequality in question. We shall start with the following observation due to Goetgheluck 1980:

Example 1.2 Let $E_k = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq x^k, 0 \leq x \leq 1\} \quad (k \geq 1)$. Then the set E is a Markov set with exponent $r = 2k$. Moreover, the exponent $2k$ is best possible.

This example had inspired W. Pawłucki and W. Pleśniak to investigate the Markov property of semianalytic and subanalytic sets, and more general, sets with polynomial cusps. Let us recall that a subset E of $\mathbb{R}^n$ is said to be *semianalytic* if for each point $x \in \mathbb{R}^n$ one can find a neighbourhood U of x and a finite number of real analytic functions f_{ij} and g_{ij} defined in U, such that

$$E \cap U = \bigcup_i \bigcap_j \{f_{ij} > 0, g_{ij} = 0\}.$$

The projection of a semianalytic set need not be semianalytic (Łojasiewicz 1964). The class of sets obtained by enlarging that of semianalytic sets to include images under the projections has been called the class of subanalytic sets. More precisely, a subset E of $\mathbb{R}^n$ is said to be *subanalytic* if for each point $x \in \mathbb{R}^n$ there exists an open neighbourhood U of x such that $E \cap U$ is the projection of a bounded semianalytic set A in $\mathbb{R}^{n+m}$, where $m \geq 0$. If $n \geq 3$, the class of subanalytic sets is essentially larger than that of semianalytic sets, the classes being identical if $n \leq 2$. The union of a locally finite family and the intersection of a finite family of subanalytic sets is subanalytic. The closure, interior, boundary and complement of a subanalytic set is still subanalytic, the last property being a (non-trivial) theorem of Gabrielov.

It is clear that the set E of Goetgheluck's example is semianalytic, whence subanalytic, while that of Zerner's example is not subanalytic, since it is too flat at the origin. It appears that the family of (fat) subanalytic sets is a subfamily of a family of sets admitting only polynomial-type cusps.

Definition 1.3 A subset E of $\mathbb{R}^n$ is said to be *uniformly polynomially cuspidal* (briefly, UPC) if one can choose constants $M > 0, m \geq 1$ and $d \in \mathbb{N}$, and a mapping $h : E \times [0,1] \to \bar{E}$ such that for each $x \in \bar{E}$, $h(x,1) = x,\ h(x,\cdot)$ is a polynomial of degree $\leq d$ and

$$\operatorname{dist}(h(x,t), \mathbb{R}^n \setminus E) \geq M(1-t)^m \quad \text{for} \quad (x,t) \in \bar{E} \times [0,1].$$

By an application of the famous Hironaka *rectilinealization theorem*, it was proved by Pawłucki and Pleśniak 1986 that

Theorem 1.4 *Every bounded subanalytic subset of* $\mathbb{R}^n$ *with* $\operatorname{int}E$ *dense in* E *is* UPC.

The family of UPC sets is essentially larger than that of subanalytic sets. A simple example of a UPC set which is not subanalytic is given by $[0,1] \times [-1,1] \setminus E$, where E is the set of Zerner's example.

The UPC sets are important from the pluripotential theory point of view, since they admit (pluricomplex) Green functions with nice continuity properties. To explain this, let us suppose that E is a compact subset of $\mathbb{C}^n$. We set

$$V_E(z) = \sup\{u(z) :\ u \in \mathcal{L}(\mathbb{C}^n),\ u|_E \leq 0\},\ z \in \mathbb{C}^n,$$

where

$$\mathcal{L}(\mathbb{C}^n) = \{u \in PSH(\mathbb{C}^n) :\ \sup_{z \in \mathbb{C}^n} [u(z) - \log(1 + |z|)] < \infty\}$$

is the *Lelong class* of plurisubharmonic functions with minimal growth. The function V_E is called the *(plurisubharmonic) extremal function* associated with E. Its upper semicontinuous regularization V_E^* is a multidimensional counterpart of the classical *Green function* for $\mathbb{C} \setminus \hat{E}$, where $\hat{E}$ is the polynomial hull of E, since by the pluripotential theory due to E. Bedford and B.A. Taylor it is a solution of the homogeneous complex *Monge-Ampère equation*, which is reduced in the one dimensional case to the *Laplace equation.* It is known (Zakharyuta 1976, Siciak 1981) that

$$V_E(z) = \sup\{\tfrac{1}{\deg p} \log |p(z)| : \ p \text{ is a polynomial with } \ \deg p \geq 1 \text{ and } \|p\|_E \leq 1\}. \tag{1}$$

In other words, $V_E = \log \Phi_E$, where Φ_E is Siciak's extremal function. Now the set E is said to have *Hölder's Continuity Property* (briefly, HCP) if there exist positive constants M and s such that

$$V_E(z) \leq M\delta^s \quad \text{as } \operatorname{dist}(z, E) \leq \delta \leq 1. \tag{HCP}$$

The importance of the class UPC is explained by the following

Proposition 1.5 (Pawłucki-Pleśniak 1986) *If E is a compact UPC subset of $\mathbb{R}^n$ with parameter m then E satisfies (HCP) with exponent $s = 1/2[m]$, where $[m] := k$ as $k - 1 < m \leq k$ with $k \in \mathbb{Z}$.*

Here and later on $\mathbb{R}^n$ is treated as a subset of $\mathbb{C}^n$ such that $\mathbb{R}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \ \Im z_j = 0, \ j = 1, \ldots, n\}$. Now we can come back to the multivariate Markov inequality. By an observation that goes back to Siciak 1967,

If E is HCP then it preserves Markov's inequality (MI_n).

Thus we have yielded a number of examples of sets admitting the multivariate Markov inequality. These are all UPC subsets of $\mathbb{R}^n$. There are, however, sets that are HCP without being UPC. Such (Cantor-type) sets were first constructed by Jonsson 1991 and Siciak 1993. The problem of whether the classical Cantor ternary set has Markov's property has appeared more difficult and a positive answer was first given by Białas and Volberg 1993 who showed that this set is even HCP. It is worth adding that there are also Cantor-type sets which do not preserve Markov's inequality and, at the same time, they are regular with respect to the (classical) Green function (Pleśniak 1990, Goetgheluck-Pleśniak 1992, Totik 1995). Up to now, the problem of whether Markov's property of E implies that E is HCP remains open. We know only that the answer is ”yes” for a class of one-dimensional Cantor-type sets (Białas-Cież 1995, Totik 1995). In general, we even do not know whether Markov's property of E implies the continuity of the Green function V_E or else non-pluripolarity of E. We recall that a subset E of $\mathbb{C}^n$ is said to be *pluripolar* if one can find a plurisubharmonic function u on $\mathbb{C}^n$ such that $E \subset \{u = -\infty\}$. However, Białas-Cież 1996 proved that any plane compact Markov set has a positive logarithmic capacity, whence it is not polar. She also proved (2000) that if E is a compact Markov subset of $\mathbb{R}$ then E is L-regular.

2 Polynomial approximation of $\mathcal{C}^\infty$ functions

By the celebrated Bernstein theorem, a function $f : I = [a, b] \subset \mathbb{R} \to \mathbb{C}$ extends to a $\mathcal{C}^\infty$ function in $\mathbb{R}$ if and only if, for each $s > 0$,

$$\lim_{k\to\infty} k^s \mathrm{dist}_I(f, \mathcal{P}_k) = 0.$$

By a standard argument, this beautiful result can be easily extended to the case of functions defined on (fat) convex compact sets in $\mathbb{R}^n$. In general, the classical proof of Bernstein's theorem does not work, since, contrary to the case of an interval in $\mathbb{R}$, there are compact sets E in $\mathbb{R}^n$ and functions $f : E \to \mathbb{R}$ such that f are $\mathcal{C}^\infty$ in $\mathrm{int}\, E$ and extend together with all their derivatives to continuous functions in E, but do not admit any $\mathcal{C}^\infty$ extension to an open neighbourhood of E. A standard example is the set $E = E_1 \cup E_2 \subset \mathbb{R}^2$, where $E_1 = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1, g(x) \le y \le 1\}$ with $g(x) = \exp(-1/x)$ as $0 < x \le 1$ and $g(0) = 0$, and $E_2 = [0, 1] \times [-1, 0]$, and the function $f(x, y) = g(x)$ if $(x, y) \in E_1$ and $f(x, y) \equiv 0$ if $(x, y) \in E_2$. The problem was solved by Pawłucki and Pleśniak 1986. In the sequel, we shall say that a subset E of $\mathbb{R}^n$ is $\mathcal{C}^\infty$ *determining* if for each function $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, if $f = 0$ on E, then for each $\alpha \in \mathbb{Z}^n_+$, $D^\alpha f = 0$ on E. It can be proved (Pleśniak 1990) that any compact Markov set in $\mathbb{R}^n$ is $\mathcal{C}^\infty$ determining.

Now, we are able to state a multivariate version of Bernstein's theorem.

Theorem 2.2 (Pawłucki-Pleśniak 1986, Pleśniak 1990) *If a compact set E in $\mathbb{R}^n$ is $\mathcal{C}^\infty$ determining then the following statements are equivalent:*

(i) E has Markov's property;

(ii) *E has the following property: there exist positive constants M and r such that for each polynomial $p \in \mathcal{P}_k$ $(k = 1, 2, \dots)$ one has*

$$|p(x)| \leq M\|p\|_E \quad \textit{if } \mathit{dist}(x, E) \leq 1/k^r;$$

(iii) *(Bernstein's Theorem) for every function $f : E \to \mathbb{R}$, if the sequence $\{\mathit{dist}_E(f.\mathcal{P}_k)\}$ is rapidly decreasing, i.e. for each $s > 0$, $k^s \mathit{dist}_E(f, \mathcal{P}_k) \to 0$ as $k \to \infty$, then f extends to a $\mathcal{C}^\infty$ function $\tilde{f}$ in $\mathbb{R}^n$.*

Here $\mathcal{P}_k$ denotes the space of polynomials of degree $\leq k$ and $\mathrm{dist}_E(f, \mathcal{P}_k) := \inf\{\|f - p\|_E : p \in \mathcal{P}_k\}$.

3 Extension of $\mathcal{C}^\infty$ functions from compact sets in $\mathbb{R}^n$

Let E be a compact set in $\mathbb{R}^n$ and let $\mathcal{C}^\infty(E)$ denote the space of all functions $f : E \to \mathbb{C}$ that can be extended to $\mathcal{C}^\infty$ functions in the whole space $\mathbb{R}^n$. We give the space $\mathcal{C}^\infty(E)$ the topology τ_Q endowed with the family of the seminorms

$$q_{K,k}(f) := \inf\{\|g\|_{K,k} : g \in \mathcal{C}^\infty(\mathbb{R}^n),\ g_{|E} = f\}, \tag{3.1}$$

where $k = 0, 1, \dots,$ K is any compact subset of $\mathbb{R}^n$, and

$$\|g\|_{K,k} := \max\{\sup |D^\alpha f(x)| :\ x \in K, |\alpha| \leq k\} \tag{3.2}$$

or, equivalently, the topology endowed with the family of the seminorms

$$q_k(f) := \inf\{\|g\|_{P,k} : g \in \mathcal{C}^\infty(\mathbb{R}^n),\ g_{|E} = f\}, \tag{3.3}$$

where P is a fixed compact cube in $\mathbb{R}^n$ such that $E \subset \mathrm{int}\, P$. Thus τ_Q is the quotient topology of the space $\mathcal{C}^\infty(\mathbb{R}^n)/\mathcal{I}(E)$, where $\mathcal{C}^\infty(\mathbb{R}^n)$ is endowed with the natural topology determined by the seminorms $\|\cdot\|_{K,k}$ and $\mathcal{I}(E) = \{f \in \mathcal{C}^\infty(\mathbb{R}^n) :\ f_{|E} = 0\}$. Since $\mathcal{C}^\infty(\mathbb{R}^n)$ is complete and since $\mathcal{I}(E)$ is a closed subspace of $\mathcal{C}^\infty(\mathbb{R}^n)$, the quotient space $\mathcal{C}^\infty(\mathbb{R}^n)/\mathcal{I}(E)$ is also complete, whence $(\mathcal{C}^\infty(E), \tau_Q)$ is a Fréchet space. If the set E is $\mathcal{C}^\infty$ determining, this space can be identified with the space of Whitney jets on E. Let us recall that a $\mathcal{C}^\infty$ *Whitney jet* on E is a vector $F = (F^\alpha)$ $(\alpha \in \mathbb{Z}_+^n)$, where each F^α is a continuous function defined on E, such that

$$|||F|||_{E,k} := \|F\|_{E,k} + \sup\{|(R_x^k F)^\alpha(y)/|x-y|^{k-|\alpha|} :\ x,y \in E,\ x \neq y,\ |\alpha| \leq k\} < \infty,$$

for $k = 0, 1, \dots$, where

$$\|F\|_{E,k} = \sup\{|F^\alpha(x)| :\ x \in E,\ |\alpha| \leq k\}$$

and

$$(R_x^K F)^\alpha(y) = F^\alpha(y) - \sum_{|\beta| \leq k - |\alpha|} (1/\beta!) F^{\alpha+\beta}(x)(y-x)^\beta.$$

Let us denote by $\mathcal{E}(E)$ the space of all $\mathcal{C}^\infty$ Whitney fields on E endowed with the topology τ_W determined by the seminorms $|||\cdot|||_{E,k}$ $(k = 0, 1, \dots)$. It is a Fréchet space. By Whitney's Extension Theorem ([66]), $F \in \mathcal{E}(E)$ if and only if there exists a $\mathcal{C}^\infty$ function f in $\mathbb{R}^n$ such that for all $\alpha \in \mathbb{Z}_+^n$, $D^\alpha f_{|E} = F^\alpha$. In particular, if E is $\mathcal{C}^\infty$ determining, the mapping $J : \mathcal{C}^\infty(E) \ni f \to J(f) = (D^\alpha g_{|E})_{\alpha \in \mathbb{Z}_+^n}$, where $g \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $g_{|E} = f$, is a linear bijection of $\mathcal{C}^\infty(E)$ onto $\mathcal{E}(E)$. Since, for a cube P such that $E \subset \operatorname{int} P$, the seminorms $|||\cdot|||_{P,k}$ and $\|\cdot\|_{P,k}$ are equivalent (*see* [66]), the linear bijection J is a continuous mapping, whence by Banach's theorem, it is a linear isomorphism.

Contrary to the case of $\mathcal{C}^k$ jets, for k finite, Whitney's proof does not yield a continuous linear operator extending jets from $\mathcal{E}(E)$ to functions in $\mathcal{C}^\infty(\mathbb{R}^n)$. Moreover, such an operator does not in general exist, which is e.g. the case when E is a single point. The problem of the existence of such an operator has a long history. Positive examples were first given by Mityagin 1961 and Seeley 1964 (case of a half-space in $\mathbb{R}^n$). Stein 1970 showed that such an operator exists if E is the closure of a domain in $\mathbb{R}^n$ whose boundary is locally of class *Lip* 1. Bierstone 1978 extended this result to the case of *Lip* α domains with $0 < \alpha \leq 1$. He also proved that an extension operator exists if E is a fat (i.e. $\overline{\operatorname{int} E} \supset E$) closed subanalytic subset of $\mathbb{R}^n$. His method is essentially based on the famous Hironaka Desingularization Theorem. Another method based on results of Vogt and Wagner concerning the splitting of exact sequences of nuclear Fréchet spaces was applied by Tidten 1979 to show the existence of an extension operator for closed sets in $\mathbb{R}^n$ admitting some polynomial-type cusps.

All the above mentioned sets are *UPC* (whence they are Markov). It appears that some restrictions concerning cuspidality of E are necessary, since Tidten 1979 proved that for the set E of Example 1.1 (which is not Markov) there is no continuous linear extension operator from $(\mathcal{C}^\infty(E), \tau_Q)$ to the space $\mathcal{C}^\infty(\mathbb{R}^2)$. However, Pawłucki and Pleśniak 1988 showed that if E is a Markov compact subset of $\mathbb{R}^n$, then one can easily construct a continuous linear operator extending $\mathcal{C}^\infty$ functions on E to $\mathcal{C}^\infty$ functions in $\mathbb{R}^n$. In order to state this result, we give the space $\mathcal{C}^\infty(E)$ a topology connected with Jackson's theorem. To this end, let us put $d_{-1}(f) := \|f\|_E$, $d_0(f) := \mathrm{dist}_E(f, \mathcal{P}_0) = \inf\{\sup_{x\in E} |f(x) - c| : \ c \in \mathbb{C}\}$ and, for $k \geq 1$,

$$d_k(f) := \sup_{l\geq 1} l^k \mathrm{dist}_E(f, \mathcal{P}_l).$$

By Jackson's theorem the functionals d_k are seminorms on $\mathcal{C}^\infty(E)$. Let us denote by τ_J the topology of $\mathcal{C}^\infty(E)$ determined by this family of seminorms. In general, it is not Fréchet. We are now in a position to state the following.

Theorem 3.1 (Pleśniak 1990) *Let E be a $\mathcal{C}^\infty$ determining compact subset of $\mathbb{R}^n$. Then the following requirements are equivalent:*

(i) E is Markov;

(iv) the space $(\mathcal{C}^\infty(E), \tau_J)$ is complete;

(v) the topologies τ_J and τ_Q for $\mathcal{C}^\infty(E)$ coincide;

(vi) there exists a continuous linear operator

$$L : (\mathcal{C}^\infty(E), \tau_J) \to \mathcal{C}^\infty(\mathbb{R}^n)$$

such that $Lf_{|E} = f$ for each $f \in \mathcal{C}^\infty(E)$.

Moreover, if E is Markov, such an operator can be defined by

$$Lf = u_1 L_1 f + \sum_{k=1}^{\infty} u_k (L_{k+1} f - L_k f), \qquad (PP)$$

where $L_k f$ is a Lagrange interpolation polynomial of f of degree k and u_k are specially chosen cut-off functions.

By Jackson's theorem, the topology τ_Q is finer than the Jackson topology τ_J. Hence by Theorem 3.1 we get

Corollary 3.2 *If E is a Markov compact subset of $\mathbb{R}^n$ then the assignement (PP) defines a continuous linear extension operator*

$$L : \ (\mathcal{C}^\infty(E), \tau_Q) \to \mathcal{C}^\infty(\mathbb{R}^n).$$

4 Markov Exponent

By Theorem 2.2, if E is a Markov compact subset of $\mathbb{R}^n$ and $f : E \to \mathbb{C}$ admits rapid uniform approximation by polynomials on E then f extends to a $\mathcal{C}^\infty$ function in $\mathbb{R}^n$. In general, the extension is done at the cost of a lost of regularity of f. It is seen by the following

Example 4.1 (Pleśniak 1994) Let $F_p = \{(x, y) \in \mathbb{R}^2 : \ x^p \leq y \leq 1, 0 < x \leq 1\}$ and let $F = \{(x, y) \in \mathbb{R}^2 : \ 0 \leq x \leq 1, -1 \leq y \leq 0\}$. Let $f(x, y) = \exp(-1/x)$, if $(x, y) \in F_p$ and $f(x, y) = 0$, if $(x, y) \in F$. Then f is $\mathcal{C}^\infty$ in $\operatorname{int} E_p$ where $E_p = F_p \cup F$ and all derivatives of f extend continuously to $\bar{E}_p$. Moreover, they admit the following *Gevrey* type estimates:

$$\|D^\alpha f\|_{E_p} \leq C^{|\alpha|}|\alpha|^{2|\alpha|}, \quad \text{for } \alpha \in \mathbb{Z}^2_+. \tag{4.1}$$

Since the set $\bar{E}_p$ is p-regular in the sense of Whitney, f can be extended to a $\mathcal{C}^\infty$ function g on $\mathbb{R}^n$(*see* Bierstone 1980). However, if $p \geq 2$, there is no open neighbourhood U of $\bar{E}_p$ such that the extension g could satisfy estimates (4.1) (with exponent 2) in U, which can be easily seen by the Mean Value Theorem.

It was shown by Pleśniak 1994 that if we know the constant r of (MI_n) then we can estimate the loss of regularity of a $\mathcal{C}^\infty$ extension of f. This motivates the following definition of *Markov's exponent* of a compact set E in $\mathbb{R}^n$:

$$\mu(E) := \inf\{r > 0 : E \text{ satisfies } (MI_n) \text{ with } r\}.$$

If E is not a Markov set, we set $\mu(E) = \infty$. By the fact that the Chebyshev polynomials are best possible for (MI_1), one can prove that if E is a compact set in $\mathbb{R}^n$ then $\mu(E) \geq 2$. In particular, if E is a fat, convex compact subset of $\mathbb{R}^n$, then by a standard argument based on inequality (MI_1), $\mu(E) = 2$. If E is a UPC compact subset of $\mathbb{R}^n$ with parameter m then by Baran 1994, $\mu(E) = 2m$.

It appears that Markov's exponent is invariant under "good" analytic mappings. More precisely, it was proved by Baran and Pleśniak 1995 that

Theorem 4.2 *If E is a compact subset of $\mathbb{R}^n$ satisfying (MI_n) with an exponent r, and f is an analytic mapping defined in a neighbourhood U of E, with values in $\mathbb{R}^n$, such that $f(E)$ is not pluripolar (in $\mathbb{C}^n$) and $\det d_x f \neq 0$ for each $x \in E$, then $f(E)$ also satisfies (MI_n) with the same exponent r as that of E.*

This result is sharp in the sense that if the assumption $\det d_x f \neq 0$ is not satisfied for all $x \in E$ then the exponent $\mu(f(E))$ may increase (Baran-Pleśniak 1995). Moreover, if we knew that Markov's property of E implies that E is not pluripolar, we could remove in the above theorem the assumption for $f(E)$ to be not pluripolar.

5 Final remarks

It is difficult to survey all ramifications of the multivariate Markov inequality, so we have concentrated only on its uniform norm version. For its L_p versions we refer the reader to papers by Bos-Milman 1995, Goetgheluck 1987 and Baran [6].

Finally, let us mention some new topic in the recent research on Markov's inequality. These are

- Markov and Bernstein-type inequalities on curves or submanifolds in $\mathbb{R}^n$ (Bos-Levenberg-Taylor 1995, Bos-Levenberg-Milman-Taylor 1995, Baran-Pleśniak 1997, Baran-Pleśniak [8],[9],[10] ;

- Markov property of Julia sets (Kosek[31],[32],[33]);

- Markov type inequalities in Banach spaces (Sarantopoulos[51], Harris [26], Munõz-Sarantopoulos[39], Baran [5]);

- Markov sets in polynomially bounded o-minimal structures (Pleśniak [49]), (Pierzchała [43]).

References

[1] M.S. Baouendi and C. Goulaouic, *Approximation of analytic functions on compact sets and Bernstein's inequality*, Trans. Amer. Math. Soc. **189** (1974), 251–261.

[2] M. Baran, *Complex Equilibrium Measure and Bernstein Type Theorems for Compact Sets in* $\mathbb{R}^n$, Proc. Amer. Math. Soc. **123 (2)** (1995), 485–494.

[3] M. Baran, *Bernstein Type Theorems for Compact Sets in* $\mathbb{R}^n$ *Revisited*, J. Approx. Theory **79 (2)** (1994), 190–198.

[4] M. Baran, *Markov inequality on sets with polynomial parametrization*, Ann. Polon. Math. **60 (1)** (1994), 69–79.

[5] M. Baran, *Polynomial Inequalities in Banach Spaces (I)*, Jagiellonian University, Cracow, preprint (2002).

[6] M. Baran, *Markov's Inequality in* L^p *Norms (I)*, Jagiellonian University, Cracow, preprint (2003).

[7] M. Baran and W. Pleśniak, *Markov's exponent of compact sets in* $\mathbb{C}^n$, Proc. Amer. Math. Soc. **123** (9) (1995), 2785–2791.

[8] M. Baran and W. Pleśniak, *Bernstein and van der Corput-Schaake type inequalities on semialgebraic curves*, Studia Math. (1997), **125** 83–96

[9] M. Baran and W. Pleśniak, *Polynomial Inequalities on Algebraic Sets*, Studia Math.**41 (3)** (2000), 209–219

[10] M. Baran and W. Pleśniak, *Characterization of compact subsets of algebraic varieties in terms of Bernstein-type inequalities*, Studia Math.**41 (3)** (2000), 221–234

[11] S.N. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné*, Mémoires de l'Académie Royale de Belgique **4** (2) (1912), 1–103.

[12] L. Białas-Cież, *Equivalence of Markov's property and Hölder continuity of the Green function for Cantor-type sets*, East Journal on Approximations **1** (2) (1995), 249–253.

[13] L. Białas-Cież, *Markov sets in $\mathbb{C}$ are not polar*, Jagiellonian University, Preprint (1996).

[14] L. Białas and A. Volberg, *Markov's property of the Cantor ternary set*, Studia Math. **104** (1993), 259–268.

[15] E. Bierstone, *Extension of Whitney fields from subanalytic sets*, Invent. Math. **46** (1978), 277–300.

[16] E. Bierstone, *Differentiable functions*, Bol. Soc. Bras. Mat. **12** (2) (1980), 139–190.

[17] E. Bierstone and P.D. Milman, *Semianalytic and subanalytic sets*, Inst. Hautes Etudes Sci. Publ. Math. **67** (1988), 5–42.

[18] L. Bos, N. Levenberg and B.A. Taylor, *Characterization of smooth, compact algebraic curves in* $\mathbb{R}^2$, In: "Topics in Complex Analysis", eds. P. Jakóbczak and W. Pleśniak, Banach Center Publications, Institute of Mathematics, Polish Academy of Sciences **31** (1995), 125–134.

[19] L. Bos, N. Levenberg, P. Milman and B.A. Taylor, *Tangential Markov Inequalities Characterize Algebraic Submanifolds of* $\mathbb{R}^N$, Indiana Univ. Math. Journal **44** (1) (1995), 115–138.

[20] L. Bos and P. Milman, *On Markov and Sobolev type inequalities on compact subsets in* $\mathbb{R}^n$, In "Topics in Polynomials in One and Several Variables and Their Applications" (Th. Rassias et al. eds.), World Scientific, Singapore (1992), 81–100.

[21] L. Bos and P. Milman, *Sobolev-Gagliardo-Nirenberg and Markov type inequalities on subanalytic domains*, Geometric and Functional Analysis **5** (6) (1995), 853–923.

[22] P. Goetgheluck, *Inégalité de Markov dans les ensembles efillés*, J. Approx. Theory **30** (1980), 149–154.

[23] P. Goetgheluck, *Polynomial Inequalities on General Subsets of* $\mathbb{R}^N$, Colloq. Math. **57** (1) (1989), 127–136.

[24] P. Goetgheluck and W. Pleśniak, *Counter-examples to Markov and Bernstein Inequalities*, J. Approx. Theory **69** (1992), 318–325.

[25] A. Goncharov, *A compact set without Markov's property but with an extension operator for $\mathcal{C}^\infty$ functions*, Studia Math. **119** (1996), 27–35.

[26] L.A. Harris, *A Bernstein-Markov theorem for normed spaces*, J. Math. Anal. Appl.**208** (1997), 476–486.

[27] A. Jonsson, *Markov's inequality on compact sets*, In: "Orthogonal Polynomials and Their Applications" (C. Brezinski, L. Gori and A. Ronveaux, eds.) (1991), 309–313.

[28] A. Jonsson, *Markov's Inequality and Zeros of Orthogonal Polynomials on Fractal Sets*, J. Approx. Theory **78** (1994), 87–97.

[29] A. Jonsson and H. Wallin, *Function Spaces on Subsets of* $\mathbb{R}^n$, Mathematical Reports Vol. 2, Part 1, Harwood Academic, London, 1984.

[30] M. Klimek, *Pluripotential Theory*, Oxford Univ. Press, London, 1991.

[31] M. Kosek, *Hölder Continuity Property of filled-in Julia sets in* $\mathbb{C}^n$, Proc. Amer. Math. Soc. (1997), **125(7)** 2029–2032

[32] M. Kosek, *Hölder Continuity Property of Composite Julia Sets*, Bull. Polish. Acad. Sci, Math.(1998), **46(4)** 391–399

[33] M. Kosek, *I*, teration of Polynomial Mappings on Algebraic SetsComplex Variables Theory Appl.(2000), **43** 187–197

[34] J. Lithner, *Comparing two versions of Markov's inequality on compact sets*, J. Approx. Theory **77** (1994), 202–211.

[35] S. Łojasiewicz, *Ensembles semianalytiques*, Inst. Hautes Etudes Sci, Bures-sur-Yvette, 1964.

[36] B. Malgrange, *Ideals of Differentiable Functions*, Oxford University Press, Bombay, 1966.

[37] G.V. Milanović and T.M. Rassias, *On Markov-Duffin-Schaeffer Inequalities*, J. Nat. Geometry **5** (1) (1994), 29–41.

[38] B. Mityagin, *Approximate dimension and bases in nuclear spaces*, Russian Math. Surveys **16** (4) (1961), 59–128.

[39] G. Muñoz and Y. Sarantopoulos, *Bernstein and Markov-type inequalities for polynomials on real Banach spaces*, Math. Proc. Cambridge Phil. Soc. (to appear).

[40] W. Pawłucki and W. Pleśniak, *Markov's inequality and $\mathcal{C}^\infty$ functions on sets with polynomial cusps*, Math. Ann. **275** (1986), 467–480.

[41] W. Pawłucki and W. Pleśniak, *Extension of $\mathcal{C}^\infty$ functions from sets with polynomial cusps*, Studia Math. **88** (1988), 279–287.

[42] W. Pawłucki and W. Pleśniak, *Approximation and extension of $\mathcal{C}^\infty$ functions defined on compact subsets of $\mathbb{C}^n$*, In: "Deformations of Mathematical Structures" (J. Ławrynowicz ed.), Kluwer Academic Publishers (1989), 283–295.

[43] R. Pierzchała, *UPC condition in polynomially bounded o-minimal structures*, J. Approx. Theory**132** (2005), 25–33.

[44] W. Pleśniak, *Compact subsets of $\mathbb{C}^n$ preserving Markov's inequality*, Mat. Vesnik **40** (1988), 295–300.

[45] W. Pleśniak, *A Cantor regular set which does not have Markov's property*, Ann. Polon. Math. **51** (1990), 269–274.

[46] W. Pleśniak, *Markov's inequality and the existence of an extension operator for $\mathcal{C}^\infty$ functions*, J. Approx. Theory **61** (1990), 106–117.

[47] W. Pleśniak, *Extension and polynomial approximation of ultradifferentiable functions in $\mathbb{R}^n$*, Bull. Soc. Roy. Sci. Liège **63** (5) (1994), 393–402.

[48] W. Pleśniak, *Remarks on Jackson's Theorem in $\mathbb{R}^N$*, East Journal on Approximations **2** (3) (1996), 301–308.

[49] W. Pleśniak, *Pluriregularity in polynomially bounded o-minimal structures*, Univ. Iagello. Acta Math.(2003),.

[50] Q.I. Rahman, G. Schmeisser, *Les inégalités de Markoff et de Bernstein*, Presses Univ. Montréal, Montréal, Québec, 1983.

[51] Y. Sarantopoulos, *Bounds on the derivatives of polynomials on Banach spaces*, Math. Proc. Cambridge Phil. Soc. (1991), **1110** 307–312.

[52] J. Schmets and M. Valdivia, *On the existence of continuous linear analytic extension maps for Whitney jets*, Institut de Mathématique, Université de Liège, Publication n^o 95.011.

[53] R. T. Seeley, *Extension of $\mathcal{C}^\infty$ functions defined on a half-space*, Proc. Amer. Math. Soc. **15** (1964), 625-626.

[54] J. Siciak, *On some extremal functions and their applications in the theory of analytic functions of several complex variables*, Trans. Amer. Math. Soc. **105** (1962), 322–357.

[55] J. Siciak, *Degree of convergence of some sequences in the conformal mapping theory*, Colloq. Math. **16** (1967), 49–59.

[56] J. Siciak, *Extremal plurisubharmonic functions in $\mathbb{C}^n$*, Ann. Pol. Math. **39** (1981), 175–211.

[57] J. Siciak, *Rapid polynomial approximation on compact sets in $\mathbb{C}^n$*, Univ. Iagello. Acta Math. **30** (1993), 145–154.

[58] J. Siciak, *Wiener's type sufficient conditions in $\mathbb{C}^N$*, ibid **35** (1997), 47-74.

[59] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.

[60] M. Tidten, *Fortsetzungen von $\mathcal{C}^\infty$-Funktionen, welche auf einer abgeschlossenen Menge in $\mathbb{R}^n$ definiert sind*, Manuscripta Math. **27** (1979), 291–312.

[61] A.F. Timan, *Theory of Approximation of Functions of a Real Variable*, Pergamon Press, Oxford-London-New York-Paris, 1963.

[62] V. Totik, *Markoff constants for Cantor sets*, Acta Sci. Math. (Szeged) **60** (1995), 715–734.

[63] J. C. Tougeron, *Idéaux de Fonctions Différentiables*, Springer-Verlag, Berlin-Heidelberg-New-York, 1972.

[64] A. Volberg, *An estimate from below for the Markov constant of a Cantor repeller*, In: "Topics in Complex Analysis", eds. P. Jakóbczak and W. Pleśniak, Banach Center Publications, Institute of Mathematics, Polish Academy of Sciences **31** 393–390.

[65] K. Wachta, *Prolongement de fonctions* $\mathcal{C}^\infty$, Bull. Polish Acad. Sci. Math. **31** (1983), 245–248.

[66] H. Whitney, *Analytic extension of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), 63–89.

[67] A. Zeriahi, *Inégalités de Markov et développement en série de polynômes orthogonaux des fonctions* $\mathcal{C}^\infty$ *et* $\mathcal{A}^\infty$, in: "Proceedings of the Special Year of Complex Analysis of the Mittag-Leffler Institute 1987-88" (ed. J.F. Fornaess), Princeton Univ. Press, Princeton New Jersey (1993), 693–701.

[68] M. Zerner, *Développement en séries de polynômes orthonormaux des fonctions indéfiniment différentiables*, C. R. Acad. Sci. Paris Sér. I **268** (1969), 218–220.